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The Uniqueness of the Cubic Lattice Graph

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ABSTRACT

A cubic lattice graph is defined as a graph G, whose vertices are the ordered triplets on n symbols, such that two vertices are adjacent if and only if they have two coordinates in common. Laskar characterized these graphs for $n > 7$ by means of five conditions. In this paper the same characterization is shown to hold for all n except for $n = 4$, where the existence of exactly one exceptional case is demonstrated.

I. INTRODUCTION

All the graphs considered in this paper are finite undirected, without loops or parallel edges. Let us define a cubic lattice graph as a graph G, whose vertices are identified with the n^3 ordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two coordinates in common. Let $d(x, y)$ denote the distance between two vertices x and y, and $\Delta(x, y)$ the number of vertices adjacent to both x and y, then a cubic lattice graph G is readily seen to have the following properties:

(P1) The number of vertices is n^3.

(P2) G is connected and regular of degree $3(n - 1)$.

(P3) If $d(x, y) = 1$, then $\Delta(x, y) = n - 2$.

(P4) If $d(x, y) = 2$, then $\Delta(x, y) = 2$.

(P5) If $d(x, y) = 2$, then there exist exactly $n - 1$ vertices z, such that $d(x, y) = 1$ and $d(y, z) = 3$.

Laskar [4] succeeded in proving that for $n > 7$ the conditions (P1)–(P5) characterize cubic lattice graphs. In the present paper we supplement her work, using a different method, by showing that, except for $n = 4$, in which case exactly one exceptional graph exists, any graph G possessing the properties (P1)–(P5) must be a cubic lattice graph. In the case of unordered triplets a similar characterization question arises and has been solved in the affirmative for $n > 16$ by Bose and Laskar [2] and for $n \leqslant 8$ by the author [1].

Notation

1. C_i denotes the cycle of length i.

2. K_i denotes the complete graph of order i.

3. By the subgraph induced by a set S of vertices in G we mean the subgraph which has S as its point-set and includes all edges between any two points in S.

4. $D_i(x)$ denotes the subgraph induced by all vertices of distance i to x.

5. $G = A_1 \cup \cdots \cup A_t$ means that the vertex set of G is the union of the *disjoint* subsets $A_1 ,..., A_t$. (There may be edges between the A_i's, however.)

II. The Main Theorem

THEOREM 1. *A graph G is a cubic lattice graph if and only if G satisfies* (P1)–(P5) *and the following property* (P6):

(P6) *For all $x \in G$ the subgraph $D_1(x)$ can be partitioned into* 3 *vertex-disjoint K_{n-1}'s.*[1]

PROOF: Necessity follows readily from the definition. To prove the sufficiency of (P1)–(P6) let us interpret cubic lattice graphs geometrically. For $n \geqslant 1$ (integral) let

$$R_n = \{(i, j, k) \mid 1 \leqslant i, j, k \leqslant n; i, j, k \text{ integral}\},$$

i.e., the set of all integral lattice points of the 3-cube with sides extending from 1 to n. Now it is evident from the definition that G is a cubic lattice graph if and only if its vertices can be identified with the lattice points in R_n, such that two vertices are adjacent if and only if they lie on a straight line parallel to a coordinate axis.

Thus in order to prove the theorem it suffices to show that the vertices of a graph G satisfying (P1)–(P6) can be arranged in R_n in the above manner.

[1] By (P3) it is clear that there exist no edges joining vertices of different K_{n-1}'s.

Let us denote the vertices of G by the natural numbers 1 to n^3 and let

$$\left\{\begin{matrix} 2,\ldots, n \\ n+1,\ldots, 2n-1 \\ 2n,\ldots, 3n-2 \end{matrix}\right\}$$ be the 3 K_{n-1}'s constituting $D_1(1)$ with numbers in the same row inducing K_{n-1}'s.

Place 1 at the spot $(1, 1, 1)$ in R_n, $\{2,\ldots, n\}$ on the lattice points $\{(i, 1, 1) \mid 2 \leqslant i \leqslant n\}$ in this order, $\{n+1,\ldots, 2n-1\}$ on $\{(1, j, 1) \mid 2 \leqslant j \leqslant n\}$, and $\{2n,\ldots, 3n-2\}$ on $\{(1,1,k) \mid 2 \leqslant k \leqslant n\}$. Next we take $D_1(2)$. Beside $\{1, 3,\ldots, n\}$ inducing a K_{n-1}, there are another $2n-2$ points $\{3n-1,\ldots, 5n-4\}$, say. Now by (P3) and (P4) it is clear that each one of $\{3n-1,\ldots, 5n-4\}$ is adjacent to exactly one of $\{n+1,\ldots, 3n-2\}$ and vice versa. Let $3n-1$ be adjacent to $n+1$, $3n$ to $n+2,\ldots, 5n-4$ to $3n-2$. Now place $\{3n-1,\ldots, 4n-3\}$ on the spots $\{(2, j, 1) \mid 2 \leqslant j \leqslant n\}$ in this order, and $\{4n-2,\ldots, 5n-4\}$ on $\{(2, 1, k) \mid 2 \leqslant k \leqslant n\}$. We have to make sure that each of the sets $\{3n-1,\ldots, 4n-3\}$ and $\{4n-2,\ldots, 5n-4\}$ induces a K_{n-1}. Consider 1, $3n-1$. By (P5) there are $n-1$ points adjacent to 1 and at distance 3 from $3n-1$. These points cannot be in the set $\{3, 4,\ldots, n\}$ or in $\{n+2,\ldots, 2n-1\}$, since they are all of distance 2 from $3n$. Hence the points $\{2n, 2n+1,\ldots, 3n-2\}$ constitute all such points. Now the $n-2$ points adjacent to both 2 and $3n-1$ clearly are not in $\{1, 3,\ldots, n\}$ by (P3). If one is in $\{4n-2,\ldots, 5n-4\}$, say $4n-2$, then $d(2n, 3n-1) = 2$, which is a contradiction.

Next we consider $D_1(3)$. Beside $\{1, 2, 4,\ldots, n\}$ there are $2n-2$ more points, let us denote them by $\{5n-3,\ldots, 7n-6\}$. As before each one of $\{5n-3,\ldots, 7n-6\}$ is adjacent to exactly one of $\{n+1,\ldots, 3n-2\}$ and vice versa. Let $5n-3$ be adjacent to $n+1,\ldots, 7n-6$ to $3n-2$. Now the situation is an in Figure 1. Vertices lying on the same straight line induce K_{n-1}'s in Figure 1.

We claim: $5n-3$ is adjacent to $3n-1,\ldots, 7n-6$ to $5n-4$. First of all by the same argument as before we conclude that each vertex of $\{5n-3,\ldots, 7n-6\}$ is adjacent to exactly one vertex of $\{3n-1,\ldots, 5n-4\}$ and conversely. Further (P5) implies there are no edges between the sets $\{3n-1,\ldots, 4n-3\}$ and $\{6n-4,\ldots, 7n-6\}$, and no edges between $\{4n-2,\ldots, 5n-4\}$ and $\{5n-3,\ldots, 6n-5\}$. Finally without loss of generality suppose $d(3n-1, 5n-3) = 2$, then $3n-1$ is adjacent to some $x \in \{5n-2,\ldots, 6n-5\}$, and $5n-3$ is adjacent to some $y \in \{3n,\ldots, 4n-3\}$. But in this case the three points x, y, $n+1$ are all adjacent to both $3n-1$ and $5n-3$, in violation of (P4). Hence our assertion was correct.

In this way one considers all sets $D_1(a)$ for $1 \leqslant a \leqslant n$ and fills the lattice points

$$\{(i, j, 1) \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\} \quad \text{and} \quad \{(i, 1, k) \mid 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant n\},$$

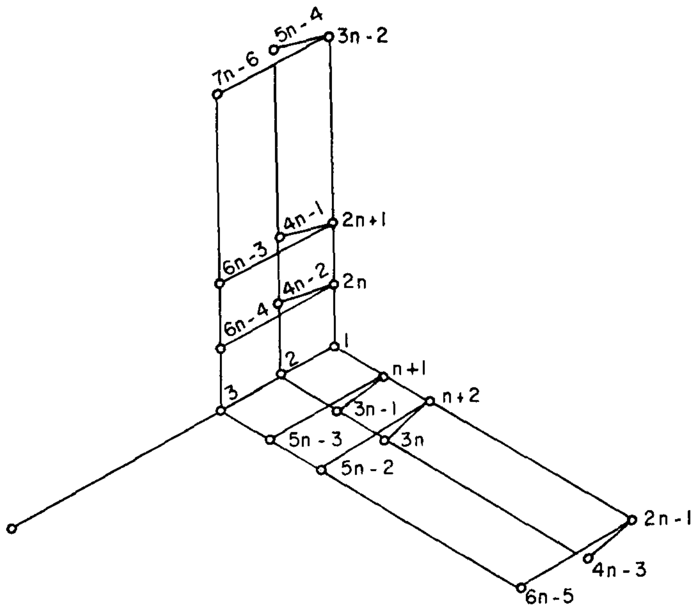

FIGURE 1

thus obtaining $2n^2 - n$ vertices. Now we turn to the sets $D_1(b)$ with $n + 1 \leqslant b \leqslant 2n - 1$. Of the points in $D_1(n + 1)\{1, n + 2,..., 2n - 1\}$ and $\{3n - 1, 5n - 3,..., 2n^2 - 3n + 3\}$ are already known to induce K_{n-1}'s. Hence the remaining points, $\{2n^2 - n + 1,..., 2n^2 - 1\}$, make up the third K_{n-1}, and each of them is adjacent to exactly one of $\{2n,..., 3n - 2\}$ and conversely, by appealing to (P4) again. W.l.o.g. suppose $2n^2 - n + 1$ is adjacent to $2n$,..., $2n^2 - 1$ to $3n - 2$, and place the set $\{2n^2 - n + 1,..., 2n^2 - 1\}$ on the spots $\{(1, 2, k) \mid 2 \leqslant k \leqslant n\}$.

In this manner we gradually fill up all the lattice points of R_n, obtaining finally all the points of G. By the above arguments it is easily seen that by means of this construction we actually arrive at a cubic lattice graph, thus proving the theorem.

REMARK 1. It is interesting to note that we made use of (P6) only at the start of the construction, when setting up $D_1(1)$. Hence we may relax condition (P6) to

(P6′) For at least one vertex $x \in G$ the subgraph $D_1(x)$ can be partitioned into 3 vertex-disjoint K_{n-1}'s.

REMARK 2. In summary we obtain the following analysis of the properties (P1)–(P6). Laskar [4] characterized the cubic lattice graphs by means of (P1), (P2), (P3), (P4), (P5). The condition $n > 7$ was used only to obtain Lemma 2.2.1 of Laskar's paper. Dowling [3] has shown that (P5) can be deduced from (P1), (P2), (P3), (P4)) and the Lemma 2.2.1. Thus, combining Laskar's characterization and Dowling's note, a cubic lattice graph can be characterized by (P1), (P2), (P3), (P4) for $n > 7$. Theorem 1 in this paper says: (P1), (P2), (P3), (P4), (P5), (P6) characterize such graphs for all n. It can be easily shown that assuming (P1), (P2), (P3), (P4), Lemma (2.2.1) in [4] is equivalent to (P6). Thus, from Dowling's observation it follows that (P5) could be deduced from (P1), (P2), (P3), (P4), and (P6). Hence Theorem 1 can be stated: (P1), (P2), (P3), (P4), (P6) characterize the cubic lattice graph. It may be noted, however, that to obtain (P5) from other conditions, (P6) cannot be replaced by the weaker (P6′).

III. A Class of Regular Graphs

To determine the structure of graphs satisfying (P1)–(P5) it seems appropriate to investigate the subgraph $D_1(x)$ where x is an arbitrary vertex of G.

For $n > 1$ (integer) let

$$\mathscr{G}_n = \{\text{all graphs } G \text{ which satisfy (Q1)–(Q3)}\},$$

(Q1) number of vertices of G is $3(n-1)$,

(Q2) G is regular of degree $n-2$,

(Q3) there are no C_4's in G except when appearing in a K_i ($i \geqslant 4$).

Remark 1. (Q1) is just condition (P2) for $D_1(x)$, (Q2) is condition (P3), and (Q3) is a consequence of condition (P4).

Remark 2. Clearly, for any n, the graph

$$G = K_{n-1} \cup K_{n-1} \cup K_{n-1}$$

is contained in $\mathscr{G}_n$. In the following we will refer to this graph as the standard graph of $\mathscr{G}_n$. If for some n we can show that the standard graph is the only member of $\mathscr{G}_n$, then in the light of Theorem 1 any graph satisfying (P1)–(P5) is necessarily a cubic lattice graph for this particular n.

Remark 3. (Q3) plainly implies that for $G \in \mathscr{G}_n$ and z an arbitrary point of G

$$D_1(z) = K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_m} \qquad \text{with} \qquad i_1 + \cdots + i_m = n - 2.$$

Each of $\mathscr{G}_2$ and $\mathscr{G}_3$ contain just a single graph, the former graph having 3 isolated points, the latter graph $G = K_2 \cup K_2 \cup K_2$, i.e., the union of 3 edges.

$\mathscr{G}_4$ apparently contains three graphs:

$$C_9, \qquad C_3 \cup C_6, \qquad K_3 \cup K_3 \cup K_3.$$

We will return to the case $n = 4$ in Section V. Until then we will assume $n > 4$.

THEOREM 2. *For $n > 4$ let G be an arbitrary member of $\mathscr{G}_n$ different from the standard graph, then G does not have any complete subgraphs K_i for $i \geqslant 4$.*

PROOF: We choose $G \in \mathscr{G}_n$ different from the standard graph (if there are any other graphs) and keep it fixed throughout the argument. Let i be the greatest integer for which a K_i in G exists. To prove the theorem we assume $i \geqslant 4$ and show that this leads to a contradiction. Now it is impossible for two points $x, y \in G$ to be both in two different K_i's at the same time, since by (Q3) any two points of these two K_i's would have to be adjacent, hence a K_{i+1} would result. Thus two K_i's intersect in at most one point.

Let z be an arbitrary vertex of G, then by Remark 3,

$$D_1(z) = K_{i_1} \cup \cdots \cup K_{i_m}, \qquad i_1 + \cdots + i_m = n - 2. \tag{1}$$

Suppose k of the i_j's equal $i - 1$, i.e., z is contained in k K_i's, then

$$k(i-1) \leqslant n - 2. \tag{2}$$

Since because of (Q3) any $y \notin D_1(z)$ can be adjacent to at most one vertex of $D_1(z)$, we have

$$k(i-1)(n-i-1) + (n-2-k(i-1))(n-i) \leqslant 3(n-1) - n + 1. \tag{3}$$

By a simple computation (3) is transformed into

$$-k(i-1) + (n-2)(n-i) \leqslant 2(n-1),$$

whence, by (2),

$$(n-2)(n-i-1) \leqslant 2(n-1) \tag{4}$$

follows.

Rearranging (4) we finally arrive at

$$n - i \leqslant 2\frac{n-1}{n-2} + 1 < 4 \qquad \text{for } n > 4. \tag{5}$$

(5) now implies $n - i = 1$ or 2 or 3 and hence together with (2)

$$k(i-1) \leqslant i + 1 \qquad \text{or} \qquad k \leqslant \frac{i+1}{i-1} < 2 \qquad \text{for } i > 3. \tag{6}$$

Thus by the hypothesis $i \geqslant 4$ we infer that an arbitrary vertex $z \in G$ is contained in at most one maximal complete subgraph K_i.

To complete the proof we treat the three cases $n - i = 1, 2, 3$ separately.

1. $n - i = 1$. In this case $G = K_{n-1} \cup R$. If for any $w \in R$ $D_1(w)$ is another K_{n-1}, then we plainly arrive at the standard graph of $\mathscr{G}_n$. Hence suppose for all $w \in R$

$$D_1(w) = K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_m} \qquad \text{with} \quad m \geqslant 2.$$

Let us choose an arbitrary $w \in R$, and x, y in distinct complete subgraphs of $D_1(w)$. Now writing

$$R = D_1(w) \cup R',$$

we conclude that at least $n - 2$ edges lead from x or y into R', and at least one from each of the other $n - 4$ vertices of $D_1(w)$ into R'. Thus we obtain the inequality

$$(n-2) + (n+4) \leqslant n - 1 \qquad \text{or} \qquad n \leqslant 5.$$

For $n = 5$ finally, $D_1(w)$ must consist of a K_2 and a K_1 (otherwise 6 edges would join $D_1(w)$ with R', violating $|R'| = 4$); suppose $K_2 = \{x, y\}$, $K_1 = \{z\}$, $R' = \{u_1, u_2, u_3, u_4\}$, and x is adjacent to u_1, y to u_2, z to u_3 and u_4. Now by (Q3) u_1 cannot be adjacent to u_2, hence the C_4 u_1, u_3, u_2, u_4 would result.

Before we go into the other two possibilities which arise according to (5), we make three general observation, all implied by (Q3):

At most one edge joins two distinct K_i's. (7)

If $w \notin K_i$, then at most one edge leads from w into K_i. (8)

If v, $w \notin K_i$ are joined to two distinct vertices of K_i, then v and w are not adjacent to each other. (9)

2. $n - i = 2$. Let s denote the number of K_i's in G, then by (6)

$$si = s(n-2) \leqslant 3(n-1) \quad \text{or} \quad s \leqslant 3 \cdot \frac{n-1}{n-2} < 4 \quad \text{for } n \geqslant 6\ (i \geqslant 4). \tag{10}$$

(a) $s = 3$. In this case $G = K_i \cup K_i \cup K_i \cup R$ with $|R| = 3$, hence by (7), (8), and (Q2)

$$3i(n-i-1) - 6 \leqslant 3 \cdot 3 \qquad \text{or} \qquad i \leqslant 5. \tag{11}$$

For $i = 5$ there are at least three points of any K_i joined to R, hence by (9) R is an independent set; but this together with (8) and (Q2) presents a contradiction.

For $i = 4$ any vertex of R is of degree 4 and, since by (8) any such vertex is adjacent to at most three points outside R, there are either two or three edges in R. Now by a similar reasoning as before a contradiction arises.

(b) $s = 2$. This time $G = K_i \cup K_i \cup R$ with $|R| = n + 1$. Let Let $R = A \cup B$, where A is the set of vertices that are adjacent to a vertex in the first K_i, then by (9) and (Q2)

$$\begin{aligned} &|A| \geqslant i - 1 = n - 3 \geqslant 3, \quad A \text{ independent set,} \\ &|B| \leqslant 4. \end{aligned} \tag{12}$$

Now since an arbitrary point of A is adjacent to at most two points outside R (by (8)), and A contains at least 3 points (by (12)), we obtain the inequality

$$3(n-4) - 3 \leqslant 4 \qquad \text{or} \qquad n \leqslant 6, \tag{13}$$

where the -3 summand on the left-hand side of (13) takes (Q3) into account.

So the only possible case left is $n = 6$ (or $i = 4$) and this case is again easily dealt with by the above arguments so as to arrive at a contradiction.

(c) $s = 1$. Here $G = K_i \cup R$ and $R = A \cup B$, where as before A is the independent set of all points that are adjacent to a vertex of K_i:

$$\begin{aligned} &|A| = i = n - 2 \geqslant 4, \\ &|B| = n + 1. \end{aligned} \tag{14}$$

Choose four points of A, then by (Q3) and (14)

$$4(n-3) - 6 \leqslant n + 1 \qquad \text{or} \qquad n \leqslant 6.$$

In the only remaining case $n = 6$, the sets A and B consist of 4 and 7 points, respectively, with 12 edges between the two sets. Now, by (9) and (Q3), A is independent and at most one point in B is adjacent to a particular pair of vertices in A, and similarly at most one point in A is adjacent to the same pair in B. As can be easily seen three possibilities arise: (A) six points in B are adjacent to exactly two points in A and the remaining point to none in A, (B) one point in B is adjacent to three vertices in A, three to two vertices, three to one vertex in A, (C) five points in B are adjacent to two vertices in A, two points to one vertex in A. Let us only discuss possibility (A) as the identical argument applies to (B) and (C) as well. Let us call the six points in B that are adjacent to A $u_1, \ldots, u_6$, the remaining one u_7. Now in order to avoid a C_4 u_7 cannot be adjacent to a pair of vertices in B that is already joined to a vertex in A. But there are $3.4 = 12$ such pairs among $u_1, \ldots, u_6$, hence only three other pairs left. As 4 edges emanate from u_7 a C_4 therefore must result.

3. $n - i = 3$. Again let s denote the number of K_i's in G, then considering the number of edges emanating from a particular K_i and taking (7) and (8) into account we arrive at

$$i(n - i - 1) \leqslant 3(n - 1) - si + (s - 1),$$

from which by a simple computation

$$s \leqslant i + 1 - \frac{(n-1)(i-3)}{i-1} = i + 1 - \frac{(i+2)(i-3)}{i-1} \leqslant 3 \qquad \text{for } i \geqslant 4$$

follows. Hence as in the case $n - i = 2$, we again have three possibilities as to $s = 1, 2, 3$, and since the argument is entirely analogous the proof has been omitted.

COROLLARY 1. *For $n > 4$ the only cases of $\mathscr{G}_n$'s which contain non-standard graphs are*

$$\begin{aligned} n &= 5 \quad \textit{with} \quad i = 2 \textit{ or } 3, \\ n &= 6 \quad \textit{with} \quad i = 3, \end{aligned} \tag{16}$$

where i denotes the order of the maximal complete subgraph.

PROOF: For $i = 3$, the statement follows instantly from (5), since the argument leading up to formula (5) applies to this case as well. For $i = 2$, the graphs in consideration are without triangles and by (Q3) without C_4's, hence have girth at least 5. Now choose $z \in G$, then $D_1(z)$ is an independent set, and we immediately obtain the inequality

$$(n - 1) + (n - 2)(n - 3) \leqslant 3(n - 1),$$

which in turn yields

$$n^2 - 7n + 8 \leqslant 0. \tag{17}$$

But (17) is easily seen to have $n = 2, 3, 4, 5$ as its only integral solutions.

COROLLARY 2. *Except for possibly* $n = 4, 5, 6$, (P1)–(P5) *characterize cubic lattice graphs.*

IV. THE CASES $n = 5$ AND 6

For $n = 5$ or 6 let G be a graph satisfying (P1)–(P5) and let x be an arbitrary vertex of G. Theorem 2 states that $D_1(x)$ does not contain any K_i's for $i \geqslant 4$. A list of all possible non-standard graphs of $\mathscr{G}_5$ and $\mathscr{G}_6$, ordered according to the number of triangles, is displayed in Figures 2 and 3.

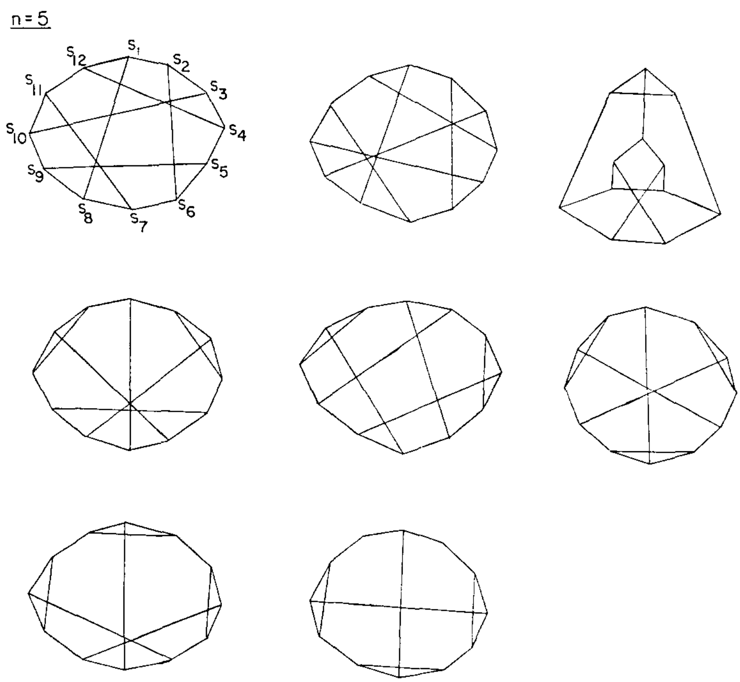

FIGURE 2. $n = 5$.

THEOREM 3. *For $n = 5$ and 6 conditions* (P1)–(P5) *characterize cubic lattice graphs.*

PROOF: In the light of Theorems 1 and 2, we have to show that the assumption $D_1(x)$ is one of the graphs of Figures 2 or 3 for some $x \in G$ leads to a contradiction. Since all the cases can be treated by an identical argument we limit this proof to the first example of Figure 2 and derive the desired contradiction.

By (P4) any $y \in D_2(x)$ is adjacent to exactly two of the s_i's which make up $D_1(x)$. If, e.g., $y \in D_2(x)$ is adjacent to s_1 and s_2, we will denote y by $s_1 s_2$; if they are several adjacent to the same pair of s_i's, we will specify them by upper indices.

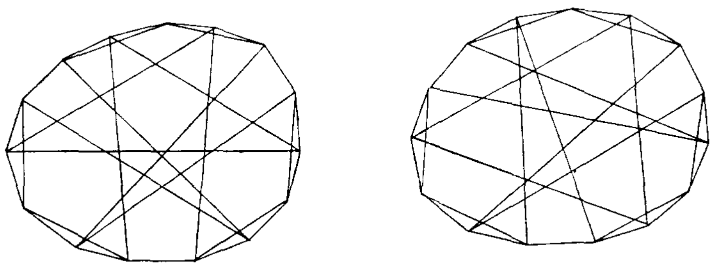

FIGURE 3. $n = 6$.

By (P2) and (P3) any s_i is adjacent to exactly 8 vertices in $D_2(x)$. In particular the points making up $D_1(s_1)$ and $D_1(s_2)$ are

$$x,\ s_2,\ s_8,\ s_{12},\ s_1s_2',\ s_1s_2'',\ s_1s_5,\ s_1s_8',\ s_1s_8'',\ s_1s_{10},\ s_1s_{12}',\ s_1s_{12}'' \tag{18}$$

for s_1, and

$$x,\ s_1,\ s_3,\ s_6,\ s_1s_2',\ s_1''s_2,\ s_2's_3,\ s_2s_3'',\ s_2s_6',\ s_2s_6'',\ s_2s_9,\ s_2s_{11} \tag{19}$$

for s_2.

Now we recall condition (P5) and apply it to s_1 and the s_i's in $D_1(x)$ which are of distance 2 to s_1, i.e.,

$$s_3,\ s_4,\ s_5,\ s_6,\ s_7,\ s_9,\ s_{10},\ s_{11}. \tag{20}$$

(P5) now states that each vertex in (20) is of distance 3 to exactly 4 points in (18) and clearly these 4 points must lie in $D_2(x)$. Taking s_9 we conclude that

$$s_9 \text{ must be of distance 3 to } s_1s_2',\ s_1s_2'',\ s_1s_{12}',\ s_1s_{12}'', \tag{21}$$

since s_9 is adjacent to s_5, s_8 and s_{10}.

Next we turn to $D_1(s_2)$ and insert all the edges in (19) which are already known to exist. We take s_2s_9 and observe that, by (P5), s_2s_9 is of distance 3 to exactly 4 points in $D_1(x)$, and from the setup of $D_1(x)$ these 4 points evidently are s_4, s_7, s_{11}, s_{12}. Hence s_2s_9 is not adjacent to s_2s_{11}, thus, (by P3), it must be adjacent to exactly three vertices among

$$\{s_1s_2', s_1s_2'', s_2s_3', s_2s_3'', s_2s_6', s_2s_6''\}.$$

Furthermore s_2s_9 cannot be adjacent to both s_2s_i' and s_2s_i'' ($i = 1, 3$, or 6) since in this case a C_4 would result; hence it must be joined to either s_1s_2' or s_1s_2'', thereby contradicting (21).

V. The Case $n = 4$

For $n = 4$ let G be a graph satisfying (P1)–(P5). As we remarked in Section III, there are two non-standard graphs in $\mathscr{G}_4$: C_9 and $C_3 \cup C_6$. Accordingly we will have to investigate these two graphs as possible candidates for $D_1(x)$ where x is a vertex of G.

Along the same lines as in Section IV it can be shown with only a little difficulty that the assumption $D_1(x) = C_9$ for some $x \in G$ leads to a contradiction. Hence we focus our attention on the case in which for some $x \in G$ (and hence for all by the remark after Theorem 1) $D_1(x) = C_3 \cup C_6$ and are going to prove

THEOREM 4. *For $n = 4$ there exists exactly one graph G beside the cubic lattice graph satisfying the conditions* (P1)–(P5).

PROOF: Let x be any vertex of G such that $D_1(x) = C_3 \cup C_6$; denote the three vertices making up C_3 by t_1, t_2, t_3; the six points constituting C_6 by $s_1, \ldots, s_6$ joined in this order. Now turning to t_1, let us determine $D(t_1)$. Employing the same notation as in Section IV we find that $D(t_1)$ has as its vertex-set x, t_2, t_3, s_1t_1, $s_2t_1, \ldots, s_6t_1$, where the first three points evidently induce a C_3, hence the last six (and by P5) apparently in this order) a C_6. We repeat this argument for t_2 and t_3, thereby constructing three C_6's in $D_2(x)$ involving the 18 vertices s_it_j ($i = 1, \ldots, 6; j = 1, 2, 3$).

Let us turn to the remaining 9 points of $D_2(x)$:

$$s_1s_2, s_1s_4, s_1s_6; \qquad s_3s_2, s_3s_4, s_3s_6; \qquad s_5s_2, s_5s_4, s_5s_6. \tag{22}$$

$D_1(s_1)$, e.g., is made up of

$$s_6, x, s_2, s_1s_2, s_1s_4, s_1s_6 \tag{23}$$

and

$$s_1t_1, s_1t_2, s_1t_3. \tag{24}$$

Since by (P5) any $y \in D_2(x)$ is of distance 3 to exactly three points in $D_1(x)$, we conclude that s_1t_1, s_1t_2, s_1t_3 are all of distance 3 to s_3, s_4, s_5, and thus cannot be adjacent to s_1s_4. Hence s_1s_4 must be joined to s_1s_2 and s_1s_6, hereby implying that the six points in (23) induce a C_6 in that order, and the three points in (24) a C_3.

Following through with this reasoning for $i = 2,..., 6$, we see (by (24)), $s_it_j \in D_2(x)$ is adjacent to $s_{i-1}t_j$, $s_{i+1}t_j$ (i mod 6 of course), s_it_{j-1}, s_it_{j+1} (j mod 3). Now since by (P2), (P4), and (P5) any $y \in D_2(x)$ is adjacent to exactly four other points of $D_2(x)$, we conclude that no edges join the 18 vertices s_it_j with the remaining 9 points in (22). Furthermore it is almost immediately clear that the vertices in (22) can be joined in only one way, without violating (P1)–(P5). Finally it is easily shown by similar arguments that $D_3(x)$ is set up in a unique fashion and is also joined to $D_2(x)$ in only one way.

In Figures 4–10 the upper triangle of the adjacency matrix of this exceptional case is displayed.

Vertices s_it_j are ordered $s_1t_1 ,..., s_1t_2 ,..., s_6t_2 , s_1t_3 ,..., s_6t_3$; points $s_1s_2 ,..., s_5s_6$ are ordered as in (22); vertices of $D_3(x)$ are denoted by $v_1 ,..., v_{27}$; 0 stands for the zero matrix.

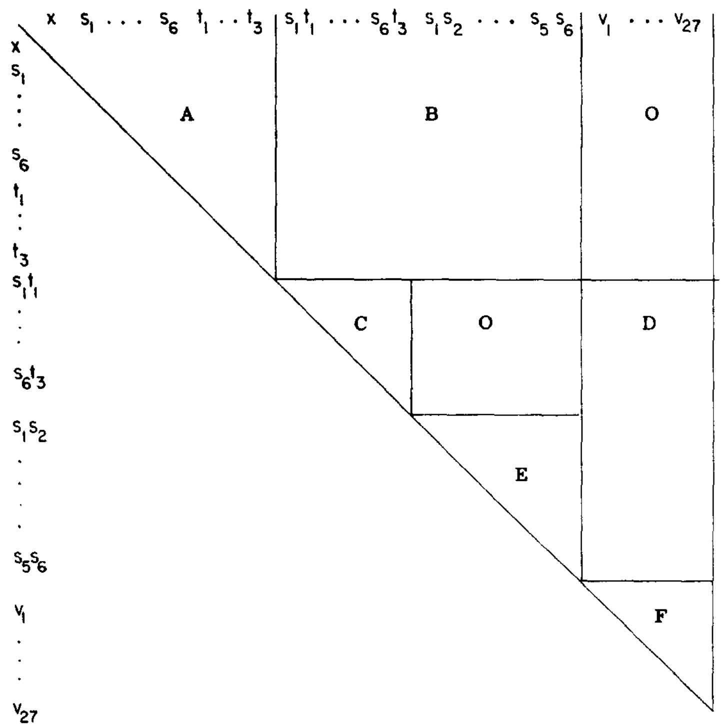

FIGURE 4

```
0 1 1 1 1 1 1 1 1 1
  0 1 0 0 0 1 0 0 0
    0 1 0 0 0 0 0 0
      0 1 0 0 0 0 0
        0 1 0 0 0 0
          0 1 0 0 0
            0 0 0 0
              0 1 1
                0 1
                  0
```

FIGURE 5. Matrix A

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 1 1 0 0 0 0 0 0
0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 1 0 0
0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1 1 1 0 0 0
0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 1 0
0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1
0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 1 0 0 1 0 0 1
1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0
```

FIGURE 6. Matrix B

```
0 1 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0 0
  0 1 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0
    0 1 0 0 0 0 1 0 0 0 0 0 1 0 0 0
      0 1 0 0 0 0 1 0 0 0 0 0 1 0 0
        0 1 0 0 0 0 1 0 0 0 0 0 1 0
          0 0 0 0 0 0 1 0 0 0 0 0 1
            0 1 0 0 0 1 1 0 0 0 0 0
              0 1 0 0 0 0 1 0 0 0 0
                0 1 0 0 0 0 1 0 0 0
                  0 1 0 0 0 0 1 0 0
                    0 1 0 0 0 0 1 0
                      0 0 0 0 0 0 1
                        0 1 0 0 0 1
                          0 1 0 0 0
                            0 1 0 0
                              0 1 0
                                0 1
                                  0
```

FIGURE 7. Matrix C

```
1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 0
0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0
0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0
0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1
0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1
1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1
```

FIGURE 8. Matrix D

```
0 1 0 0 1 0 1 0 1
  0 1 0 1 0 0 1 0
    0 1 0 1 0 1 0
      0 0 1 1 1 0
        0 1 0 0 1
          0 0 0 1
            0 1 1
              0 0
                0
```

FIGURE 9. Matrix E

```
0 1 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0
  0 1 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0
    0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1
      0 1 1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0
        0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0
          0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0
            0 1 1 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0
              0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0
                0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0
                  0 1 1 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0
                    0 1 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0
                      0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0
                        0 1 1 1 0 0 0 0 0 0 0 0 1 0 0
                          0 1 0 1 0 0 0 0 0 0 0 0 1 0
                            0 0 0 1 0 0 0 0 0 0 0 0 1
                              0 1 1 0 0 0 0 0 0 1 0 0
                                0 1 0 0 0 0 0 0 0 1 0
                                  0 0 0 0 0 0 0 0 0 1
                                    0 1 1 1 0 0 1 0 0
                                      0 1 0 1 0 0 1 0
                                        0 0 0 1 0 0 1
                                          0 1 1 0 0 0
                                            0 1 0 0 0
                                              0 0 0 0
                                                0 1 1
                                                  0 1
                                                    0
```

FIGURE 10. Matrix F

REFERENCES

1. M. AIGNER, A Characterization Problem in Graph Theory, *J. Combinatorial Theory* **6** (1968), 45–55.
2. R. C. BOSE AND R. LASKAR, A Characterization of Tetrahedral Graphs, *J. Combinatorial Theory* 3 (1967), 366–385.
3. T. A. DOWLING, Note on "A Characterization of Cubic Lattice Graphs," *J. Combinatorial Theory* **5**, 425–426.
4. R. LASKAR, A Characterization of Cubic Lattice Graphs, *J. Combinatorial Theory* **3** (1967), 386–401.